HW 2 out

Due: 10/30/25

Recap

Real Spectral Thm: $\varphi : V \longrightarrow V$ $\quad \varphi = \varphi^*$

o.n. basis of eigenvectors $v_1, \dots, v_n$

$\varphi(v_i) = \lambda_i \cdot v_i \qquad \lambda_i \in \mathbb{R}$

Rayleigh Quotients: $R_\varphi(v) = \dfrac{\langle v, \varphi(v)\rangle}{\langle v, v\rangle}$

$\lambda_1 = \max\limits_{v \neq 0} R_\varphi(v) \qquad . \qquad \lambda_n = \min\limits_{v\neq 0} R_\varphi(v)$

Positive semidefinite operators: Self-adjoint $\varphi : V \to V$

- $R_\varphi(v) \geq 0 \quad \forall\, v \neq 0$
- $\lambda_1 \geq \dots \geq \lambda_n \geq 0$
- $\varphi = \alpha^* \alpha$ for some $\alpha : V \to V$

Singular Value Decomposition (SVD)

$\varphi : V \to V$ self-adjoint

- Real Spectral Theorem
- Orthonormal eigenbasis
- Rayleigh quotients

What about general $\varphi : V \to W$?

Extending to other transformations

Let $\varphi : V \to W$ be any linear transformation. Then

$$\varphi^* \varphi : V \longrightarrow V \quad \text{and} \quad \varphi \varphi^* : W \longrightarrow W$$

$\varphi^*(\varphi(\dots))$

are self-adjoint, positive semidefinite linear operators with the same non-zero eigenvalues.

<u>Proof</u>: $\varphi^* \varphi(v) = \lambda \cdot v \Rightarrow \varphi \varphi^* \varphi(v) = \lambda \cdot \varphi(v)$
(for $v \neq 0$)

Also $\varphi(v) \neq 0$ since $\varphi^* \varphi(v) = \lambda \cdot v \neq 0$ $(\lambda \neq 0, v \neq 0)$

$\therefore$ $\varphi(v)$ is an eigenvector of $\varphi\varphi^*$

Similarly for $\varphi\varphi^*(v) = \lambda \cdot v$, $\varphi^*(v)$ is an eigenvector of $\varphi^*\varphi$.

Orthonormal eigenvectors

- Let $\varphi^*\varphi : V \to V$ have non-zero eigenvalues $\sigma_1^2 \geq \dots \geq \sigma_r^2 > 0$ with orthonormal eigenvectors $v_1, \dots, v_r$

Then $w_i = \dfrac{\varphi(v_i)}{\sigma_i}$ for $i \in [r]$

are orthonormal eigenvectors of $\varphi\varphi^* : W \to W$ with same eigenvalues $\sigma_1^2 \geq \dots \geq \sigma_r^2 > 0$.

Proof: $\varphi\varphi^*(w_i) = \dfrac{1}{\sigma_i} \cdot \varphi\varphi^*\varphi(v_i) = \dfrac{1}{\sigma_i} \cdot \sigma_i^2 \cdot \varphi(v_i) = \sigma_i^2 \cdot w_i$

$$\langle w_i, w_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle \varphi(v_i), \varphi(v_j) \rangle = \frac{1}{\sigma_i \sigma_j} \langle \varphi^*\varphi(v_i), v_j \rangle$$

$$= \frac{\sigma_i}{\sigma_j} \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

Ex: Let $V_\lambda = \{ v \in V \mid \varphi^*\varphi(v) = \lambda \cdot v \}$ $W_\lambda = \{ w \in W \mid \varphi\varphi^*(w) = \lambda \cdot w \}$.

For all $\lambda \neq 0$ $\dim(V_\lambda) = \dim(W_\lambda)$.

Understanding the action of φ

$$\varphi : V \longrightarrow W$$

$\dim(V) = n$ $\qquad$ $\dim(W) = m$

o.n. basis of eigenvecs for $\varphi^*\varphi$: $v_1, \dots, v_r, v_{r+1}, \dots, v_n$

$w_1 = \dfrac{\varphi(v_1)}{\sigma_1}, \dots, w_r = \dfrac{\varphi(v_r)}{\sigma_r}, w_{r+1}, \dots, w_m$

$\mathrm{rank}(\varphi) = ?$

$$\varphi(v_{r+1}) = 0 \quad \left(\|\varphi(v_{r+1})\|^2 = \langle v_{r+1}, \varphi^*\varphi(v_{r+1})\rangle = \langle v_{r+1}, 0\rangle = 0 \right)$$

$$v = \sum_{i=1}^{n} c_i \cdot v_i = \sum_{i=1}^{n} \langle v, v_i\rangle \cdot v_i$$

$$\varphi(v) = \sum_{i=1}^{r} \langle v, v_i\rangle \cdot \varphi(v_i) + \cancel{\sum_{i=r+1}^{n} \langle v, v_i\rangle \cdot \varphi(v_i)} \quad 0$$

$$\sum_{i=1}^{r} \langle v, v_i\rangle \cdot \sigma_i \cdot w_i$$

$$w = \sum c_j' \cdot w_j = \sum \langle w, w_j\rangle \cdot w_j$$

$$\varphi^*(w) = \sum_{i=1}^{r} \langle w, w_i\rangle \cdot \sigma_i \cdot v_i$$

$\hookrightarrow$ Ex.

Outer Products

For $v \in V$, $w \in W$, let $|w\rangle\langle v|$ denote the following linear transformation from V to W

$$|w\rangle\langle v| (u) = \langle v, u\rangle \cdot w$$

Then $\varphi : V \longrightarrow W$ with singular values / vectors

$\sigma_1 \cdots \sigma_r$ $v_1 \cdots v_r$

$w_1 \cdots w_r$

can be written as

singular values

left singular vectors

$$\varphi = \sum_{i=1}^{r} \sigma_i \, |w_i\rangle\langle v_i|$$

right singular vectors

$$\varphi(u) = \sum_{i=1}^{r} \sigma_i \cdot \langle v_i, u\rangle \cdot w_i$$

SVD practice

Ex: Prove that $\text{rank}(|v\rangle\langle w|) = 1$

Ex: Prove that $\varphi^* = \sum_{i=1}^{r} \sigma_i |v_i\rangle\langle w_i|$

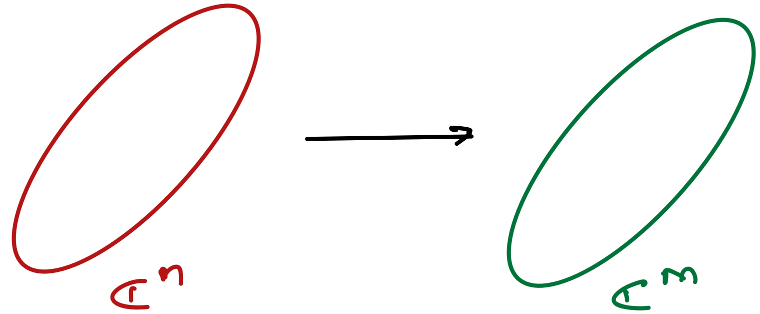
Ex: Prove that $\text{rank}(\varphi) = \text{rank}(\varphi^*) = r$

Ex: For self-adjoint $\varphi : V \to V$ with eigenvalues $\lambda_1 \dots \lambda_n$, orthonormal eigenvectors $v_1 \dots v_n$

Prove that $\varphi = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle v_i|$

SVD for matrices

$A \in \mathbb{C}^{m \times n} \quad \equiv \quad A :$

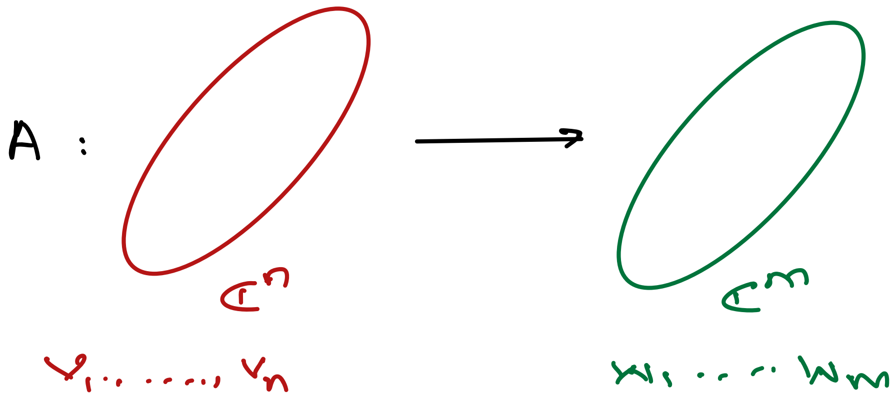


$v_1, \ldots \ldots, v_n \qquad w_i = \dfrac{Av_i}{\sigma_i}$

$$|w_i\rangle\langle v_i|\,(u) = (v_i^* u)\cdot w_i = w_i v_i^* u$$

$$A = \sum_{i=1}^{r} \sigma_i \, w_i \cdot v_i^*$$

SVD : matrix form



$$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$$

$$A = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_r \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \begin{bmatrix} \text{---} & v_1^* & \text{---} \\ & \vdots & \\ \text{---} & v_r^* & \text{---} \end{bmatrix}$$

columns form o.n. basis   rows form o.n. basis

$$A = \begin{bmatrix} | & & | & & | \\ w_1 & \cdots & w_r & \cdots & w_m \\ | & & | & & | \end{bmatrix} \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} \text{---} & v_1^* & \text{---} \\ & \vdots & \\ \text{---} & v_r^* & \text{---} \\ & & \\ \text{---} & v_n^* & \text{---} \end{bmatrix}$$

"Unitary" matrices